THE PROBLEM OF JET FLOW PAST A CONTOUR PERFORMING SMALL OSCILLATIONS

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PMM Vol.28, № 3, 1964, pp.567-571

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(Received February 16, 1964)

The boundary condition of Gurevich [1] is restated with greater accuracy for the velocity potential of perturbed flow on a solid contour, performing small oscillations and past which a stream of an ideal incompressible fluid flows with separation.

1. The problem of jet flow over a contour performing small oscillations was examined by Gurevich and Khaskind [2] in 1953. A different approach to the solution of this problem was used in 1955 by Woods [3] and then subsequently developed in [4 and 5] and other works.

Gurevich and Khaskind represented the complex potential W of the unsteady flow in the form $W = w_0 + w$, where w_0 is the complex potential of steady flow and w is the complex potential of perturbed flow, and formulated the boundary value problem for the determination of w. Using Lagrange's integral and the condition of constant pressure they derived the boundary condition for w on free surfaces. Assuming then that the normal velocity v_n of the perturbed flow on the oscillating contour is known and considering that the function $z(u)$, which reflects the flow region on a canonical region, is invariant, they established that the following relationship holds on the contour

$$\operatorname{Im} \frac{dw}{du} = -v_n \left| \frac{dz}{du} \right| \tag{1.1}$$

The upper half-plane is taken as the canonic region.

If it is assumed, as it was done in [2], that v_n is a harmonic function of time t, then the problem of determining $w(u,t)$ can be reduced to the determination of some analytical function of u in the upper half-plane. This function of u is related in a definite fashion to $w(u,t)$ through its known boundary values on the substantial axis of the plane u.

In the problem of harmonic oscillations of a flat plate it was assumed that $v_n = U_n(t)$, where $U_n(t)$ is the normal velocity of points of the oscillating plate.

Gurevich [1] assumed U_n, the normal velocity of points on the contour and $\beta(t)$, the small deflation angle of the contour as known. He established that

$$U_n = -\beta V_0 + \Delta V_{0n} + v_n \tag{1.2}$$

where ΔV_{0n} is the projection of velocity of the steady flow on the external normal to the stationary contour. ΔV_{0n} is brought about by displacement of the contour. V_0 is the velocity of the steady flow relative to the contour.

If everywhere

$$\Delta V_{0n} = \frac{\partial V_{0n}}{\partial n}\eta = \frac{\partial V_0}{\partial s}\eta \qquad \left(\eta = \int U_n dt\right)$$

where η is the displacement of contour points in the direction of normals, then

$$v_n = U_n - \frac{\partial V_0}{\partial s}\eta + \beta V_0 \tag{1.3}$$

2. Let us derive the boundary conditions for $\partial\varphi/\partial n$ on the contour assuming that the vector of progressive oscillations $\alpha(t)$ and the angle of deflection of the contour $\beta(t)$ with respect to its principal position are given.

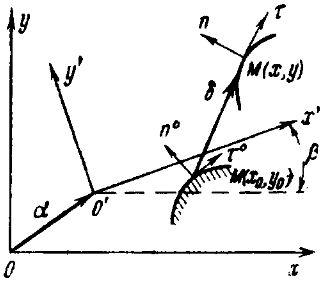

Fig. 1.

We designate through $\mathbf{n}^0$ and τ^0 respectively, the unit vectors of the external normal (directed towards the inside of the fluid) and the tangent to the stationary contour; n and τ are the normal and the tangent with respect to the oscillating contour at the instant of time under observation. The unit vectors of the tangent and the normal form a right-hand side system of Fig.1. If x_0 and y_0 are coordinates of some point M of the contour at its principal position, x and y are coordinates of the same point at the moment t in the stationary system of coordinates, x' and y' in the system of coordinates which is connected with the moving contour, then

$$x' = x_0, \qquad x' = (x - \alpha_x)\cos\beta + (y - \alpha_y)\sin\beta$$

$$y' = y_0, \qquad y' = -(x - \alpha_x)\sin\beta + (y - \alpha_y)\cos\beta$$

For small α and β we obtain, discarding small quantities of orders higher than the first

$$x' = x - \delta_x(x, t), \qquad y' = y - \delta_y(y, t) \tag{2.1}$$

Here $\delta_x = \alpha_x - y\beta$ and $\delta_y = \alpha_y + x\beta$ are the components of the displacement vector $\boldsymbol{\delta}$ of contour points; so that $\boldsymbol{\delta} = \boldsymbol{\alpha} + \boldsymbol{\gamma}$, where α corresponds to the displacements of contour points due to progressive oscillations, and $\boldsymbol{\gamma}$ corresponds to displacement of contour points due to rotation of the contour. With accuracy to small terms of first order

$$x\beta = (x_0 + \alpha_x - y\beta)\beta \approx x_0\beta, \qquad y\beta = (y_0 + \alpha_y + x\beta)\beta \approx y_0\beta$$

$$\delta_x = \alpha_x - y_0\beta \;\; \delta_y = \alpha_y + x_0\beta \tag{2.2}$$

The velocity of perturbed fluid flow at the point (x, y) is apparently equal to $\mathbf{V}(x, y) + \nabla\varphi$, where $\mathbf{V}(x, y)$ is the velocity in the steady flow.

Let $F(x, y, t) = 0$ be the equation of the solid contour in the stationary system of coordinates. The condition of no fluid flow through the contour has the form

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\mathbf{V} + \nabla\varphi)\nabla F = 0$$

or

$$\frac{\partial F}{\partial x_0}\frac{\partial x'}{\partial t} + \frac{\partial F}{\partial y_0}\frac{\partial y'}{\partial t} + (\mathbf{V} + \nabla\varphi)\left[\mathbf{i}\left(\frac{\partial F}{\partial x_0}\frac{\partial x'}{\partial x} + \frac{\partial F}{\partial y_0}\frac{\partial y'}{\partial y}\right) + \mathbf{j}\left(\frac{\partial F}{\partial x_0}\frac{\partial x'}{\partial y} + \frac{\partial F}{\partial y_0}\frac{\partial y'}{\partial y}\right)\right] = 0$$

After simple transformations utilizing (2.1) the last equation can be brought into the form

$$-\frac{\partial\boldsymbol{\delta}}{\partial t}\nabla F+(\mathbf{V}+\nabla\varphi)\left[\nabla F-\beta\left(\frac{\partial F}{\partial y_0}\mathbf{i}-\frac{\partial F}{\partial x_0}\mathbf{j}\right)\right]=0 \qquad \left(\nabla F=\frac{\partial F}{\partial x_0}\mathbf{i}+\frac{\partial F}{\partial y_0}\mathbf{j}\right) \quad (2.3)$$

The boundary condition (2.3) is fulfilled at the moment t on an oscillating contour on which $\mathbf{V}(x, y) = \mathbf{V}(x' + \delta_x, y' + \delta_y)$. Initially we shall assume that the function $\mathbf{V}(x', y')$ is continuous everywhere on the contour and has continuous derivatives up to and including the second. Then, considering the displacements of the contour to be small and limiting the expansion of $\mathbf{V}(x, y)$ in powers of $\boldsymbol{\delta}$ terms of the first order of smallness, we obtain

$$\mathbf{V}(x, y) = \mathbf{V}_0 + (\boldsymbol{\delta}\nabla)\mathbf{V}_0 \quad (2.4)$$

Here $\mathbf{V}_0$ is the velocity relative to the contour for $x = x'$ and $y = y'$, i.e. for the principal position of the contour. Substituting (2.4) into (2.3) and utilizing the boundary condition $\mathbf{V}_0\nabla F = 0$, we find

$$-\frac{\partial\boldsymbol{\delta}}{\partial t}\nabla F+[(\boldsymbol{\delta}\nabla)V_0+\nabla\varphi]\left[\nabla F-\beta\left(\frac{\partial F}{\partial y_0}\mathbf{i}-\frac{\partial F}{\partial x_0}\mathbf{j}\right)\right]-\beta\mathbf{V}_0\left(\frac{\partial F}{\partial y_0}\mathbf{i}-\frac{\partial F}{\partial x_0}\mathbf{j}\right)=0$$

Since $\mathbf{V}_0(\nabla\boldsymbol{\delta}) = 0$, $(\mathbf{V}_0\nabla)\boldsymbol{\delta} = \beta(V_{0x}\mathbf{j} - V_{0y}\mathbf{i})$ and according to the continuity equation $\nabla V_0 = \operatorname{div} V_0 = 0$, then from vector identity

$$(\mathbf{V}_0\nabla)\boldsymbol{\delta} - (\boldsymbol{\delta}\nabla)\mathbf{V}_0 = \nabla\times(\boldsymbol{\delta}\times\mathbf{V}_0) - \boldsymbol{\delta}(\nabla\mathbf{V}_0) + \mathbf{V}_0(\nabla\boldsymbol{\delta})$$

we have

$$(\boldsymbol{\delta}\nabla)\mathbf{V}_0 = -\nabla\times(\boldsymbol{\delta}\times\mathbf{V}_0) + \beta(V_{0x}\mathbf{j} - V_{0y}\mathbf{i})$$

And with accuracy to small terms of the first order

$$\nabla\varphi\nabla F = \left(\frac{\partial\boldsymbol{\delta}}{\partial t} + \nabla\times(\boldsymbol{\delta}\times\mathbf{V}_0)\right)\nabla F \quad (2.5)$$

In the derivation of this equation an idea of Timman and Newman [6], who examined the problem of small oscillations of a body moving with constant velocity, was utilized.

The right-hand side of Equation (2.5) turns out to be a small quantity of the order of δ, if $\partial\delta/\partial t \sim O(\delta)$. Discarding small terms of highest order in the left-hand side of the equation, it can be assumed that $\sim$ is evaluated at $x = x'$ and $y = y'$, i.e. on the contour at its principal position. The last conclusion is justified in the case where $\partial\delta/\partial t$ has a finite value, because here $\sim$ will have the order of $\partial\delta/\partial t$ and small quantities in both sides of (2.5) can be neglected. Consequently

$$\frac{\partial\varphi}{\partial n^\circ} = \left(\frac{\partial\boldsymbol{\delta}}{\partial t} + \nabla\times(\boldsymbol{\delta}\times\mathbf{V}_0)\right)\mathbf{n}^\circ \quad (2.6)$$

We have

$$\nabla\times(\boldsymbol{\delta}\times\mathbf{V}_0) = \nabla\times[(\delta_{\tau^\circ}\boldsymbol{\tau}^\circ + \delta_{n^\circ}\mathbf{n}^\circ)\times V_{0\tau^\circ}\boldsymbol{\tau}^\circ] =$$
$$= -\nabla\times\delta_{n^\circ}V_{0\tau^\circ}\mathbf{k} = \frac{\partial}{\partial x}(\delta_{n^\circ}V_{0\tau^\circ})\,\mathbf{i} - \frac{\partial}{\partial y}(\delta_{n^\circ}V_{0\tau^\circ})\,\mathbf{j}$$

Here $\mathbf{k}$ is the unit vector perpendicular to the plane xy. For steady flow the velocity vector on the contour is oriented along the tangent to the contour, therefore the index τ_0 in $V_{0\tau^\circ}$ will be omitted below. Further,

$$(\nabla\times(\boldsymbol{\delta}\times\mathbf{V}_0))\,\mathbf{n}^\circ = \frac{\partial}{\partial x}(\delta_{n^\circ}V_0)\cos(n^\circ, y) - \frac{\partial}{\partial y}(\delta_{n^\circ}V_0)\cos(n^\circ, x) =$$
$$= -\frac{\partial}{\partial s}(\delta_{n^\circ}V_0) = -\delta_{n^\circ}\frac{\partial V_0}{\partial s} + V_0\beta + \frac{\delta_{\tau^\circ}V_0}{R}$$

Here R is the radius of curvature of the contour. If the last relationship is substituted into (2.6), we will have

$$\frac{\partial\varphi}{\partial n^\circ}=\frac{\partial\delta_{n^\circ}}{\partial t}-\delta_{n^\circ}\frac{\partial V_0}{\partial s}+V_0\beta+\frac{V_0\delta_{\tau^\circ}}{R} \tag{2.7}$$

In view of the fact that all quantities in Equation (2.7) refer to the stationary contour, the index denoting this will also be omitted in what follows.

If the radius of curvature of the contour is sufficiently great, so that $R^{-1}\sim O(\delta)$ and $R^{-1}V\delta_\tau\sim O(\delta^2)$, then the last term of the right-hand side of Equation (2.7) can be neglrcted, then

$$\frac{\partial\varphi}{\partial n}=\frac{\partial\delta_n}{\partial t}-\delta_n\frac{\partial V}{\partial s}+V\beta \tag{2.8}$$

Since $\delta=\alpha+\beta$, then

$$\delta_n=\alpha_n(t)-\frac{1}{2}\beta(t)\frac{\partial(x^2+y^2)}{\partial s},\qquad \delta_\tau=\alpha_\tau(t)+\beta(t)\left(x\frac{\partial y}{\partial s}-y\frac{\partial x}{\partial s}\right)$$

$$\frac{\partial\delta_n}{\partial t}=\frac{\partial\alpha_n}{\partial t}-\frac{1}{2}\beta^\cdot\frac{\partial(x^2+y^2)}{\partial s}\qquad\left(\beta^\cdot=\frac{\partial\beta}{\partial t}\right) \tag{2.9}$$

$$\frac{\partial\varphi}{\partial n}=\frac{\partial\alpha_n}{\partial t}-\alpha_n\frac{\partial V}{\partial s}+\frac{1}{2}\frac{\partial(x^2+y^2)}{\partial s}\left(\beta\frac{\partial V}{\partial s}-\beta^\cdot\right)+\frac{V}{R}\left[\alpha_\tau+\beta\left(x\frac{\partial y}{\partial s}-y\frac{\partial x}{\partial s}\right)\right]+V\beta$$

From this equation it follows that in case of purely progressive oscillations of the contour

$$\frac{\partial\varphi}{\partial n}=\frac{\partial\alpha_n}{\partial t}-\alpha_n\frac{\partial V}{\partial s}+\alpha_\tau\frac{V}{R} \tag{2.10}$$

If the contour performs rational oscillations only, then

$$\frac{\partial\varphi}{\partial n}=\frac{1}{2}\frac{\partial(x^2+y^2)}{\partial s}\left(\beta\frac{\partial V}{\partial s}-\beta^\cdot\right)+\frac{V\beta}{R}\left(R+x\frac{\partial y}{\partial s}-y\frac{\partial x}{\partial s}\right) \tag{2.11}$$

Since in the derivation of the boundary condition (2.9) an assumption about the continuity of the function $\mathbf{V}(x,y)$ and of all its derivatives up to and including.the second derivative was utilized, condition (2.9) is applicable only to problems of jet flow past a certain class of obstacles. To this class belong smooth contours with finite curvature of jets at the points of separation. It is apparent, that condition (2.9) is also correct for small oscillatins of smooth closed contours surrounded by continuous fluid flow. In these cases $\mathbf{V}(x,y)$ will be an analytic function of coordinates of contour points. From Equation (2.11) in particular follows the obvious boundary value $\partial\varphi/\partial n=0$ on the circumference revolving about its center.

3. In the general case of flow with jet separation around obstacles of arbitrary shape, the Taylor's theorem for determination of increase of the function $\mathbf{V}(x,y)$ cannot be used. For example, in the simplest problem of symmetrical jet flow past a flat plate the function $\mathbf{V}(x,y)$ will be analytic everywhere, with the exception of separation points of the jet where $\partial V/\partial S$ becomes infinity. Therefore in such cases instead of the expansion (2.4) it can only be written

$$\mathbf{V}(x,y)=\mathbf{V}_0+\triangle\mathbf{V}_0$$

After making the necessary computations analogous to those which were made in the derivation of condition (2.7), we obtain

$$\frac{\partial\varphi}{\partial n}=\frac{\partial\delta_n}{\partial t}-\Delta V_n-\beta V \tag{3.1}$$

For the computation of ΔV_n the method which is examined for the problem of symmetrical jet flow past a plate, performing small progressive displacements in the direction perpendicular to the plate, can be used.

Having equations of [1]

$$z = \frac{iQ}{V_\infty}[2u + u\sqrt{1-u^2} + \sin^{-1} u], \qquad V_\infty \frac{dz}{dw} = \frac{1}{\chi} = i\,\frac{\sqrt{1-u^2}+1}{u}$$

we find

$$z(\chi) = \frac{iQ}{V_\infty}\left[\frac{2i\chi(3-\chi^2)}{(1-\chi^2)^2} - i\ln\frac{1-\chi}{1+\chi}\right] \tag{3.2}$$

Using Taylor's formula

$$z(\chi+\Delta\chi) = z + \Delta z = z(\chi) + \frac{\partial z}{\partial\chi}\Delta\chi + \frac{1}{2}\frac{\partial^2 z}{\partial\chi^2}(\Delta\chi)^2 + \ldots$$

in which $\Delta\chi$ is considered to be small and we limit ourselves to terms of second order of smallness. We obtain

$$\Delta x + i\Delta y = -\frac{8Q}{V_\infty}\frac{1+\chi^2}{(1-\chi^2)^3}(\Delta V_x - i\Delta V_y) - \frac{16Q}{V_\infty}\frac{\chi(2+\chi^2)}{(1-\chi^2)^4}(\Delta V_x - i\Delta V_y)^2$$

On the plate $V_x = 0$ and $\chi = -iV_y$, therefore

$$\Delta x = -\frac{8Q}{V_\infty}\frac{1-V_y^2}{(1+V_y^2)^3}\Delta V_x + \frac{32Q}{V_\infty}\frac{V_y(2-V_y^2)}{(1+V_y^2)^4}\Delta V_x\Delta V_y \tag{3.3}$$

$$\Delta y = \frac{8Q}{V_\infty}\frac{1-V_y^2}{(1+V_y^2)^3}\Delta V_y + \frac{16Q}{V_\infty}\frac{V_y(2-V_y^2)}{(1+V_y^2)^4}[(\Delta V_x)^2 - (\Delta V_y)^2] \tag{3.4}$$

Assuming $\Delta y = 0$, we find from (3.4)

$$\Delta V_y = \frac{1}{4}\frac{1-V_y^4}{V_y(2-V_y^2)} \pm \left[\frac{1}{16}\frac{(1-V_y^4)^2}{V_y^2(2-V_y^2)^2} + (\Delta V_x)^2\right]^{1/2}$$

and substitute this in (3.3). Since for $\Delta V_x = 0$, ΔV_y is also equal to 0, it is necessary to take the minus sign in front of the root in the last equation.

We obtain

$$\Delta x = -\frac{32Q}{V_\infty}\frac{V_y(2-V_y^2)}{(1+V_y^2)^4}\left[\frac{1}{16}\frac{(1-V_y^4)^2}{V_y(2-V_y^2)^2} + (\Delta V_x)^2\right]^{1/2}\Delta V_x \tag{3.5}$$

Solving (3.5) with respect to ΔV_x, we find

$$(\Delta V_x)^2 = -\frac{B}{2} \pm \left[\frac{B^2A^2 + 4(\Delta x)^2}{4A^2}\right]^{1/2} \tag{3.6}$$

Here

$$A = \frac{32Q}{V_\infty}\frac{V_y(2-V_y^2)}{(1+V_y^2)^4}, \qquad B = \frac{1}{16}\frac{(1-V_y^4)^2}{V_y^2(2-V_y^2)^2}$$

It is apparently necessary to take the plus sign in front of the root in (3.6). Then

$$\Delta V_x = \pm\frac{2\Delta x}{\sqrt{2A(AB + \sqrt{A^2B^2 + 4(\Delta x)^2}}} = \pm\left[\frac{-BA + \sqrt{B^2A^2 + 4(\Delta x)^2}}{2A}\right]^{1/2} \tag{3.7}$$

Since the signs of Δx and ΔV_x must be opposite, it is necessary to take the minus sign in front of the root in (3.7), consequently

$$\Delta V_x = -\frac{V_\infty}{4\sqrt{2}Q}\frac{(1+V_y^2)^3}{1-V_y^2}\left\{1+\left[1+\left(\frac{V_\infty}{Q}\frac{V_y(1+V_y^2)^2(2-V_y^2)}{(1-V_y^2)^2}\Delta x\right)^2\right]^{1/2}\right\}^{-1/2}\Delta x \qquad (3.8)$$

For small values V_y it follows from (3.8) that

$$\Delta V_x = \frac{-V_\infty(1+V_y^2)^3\Delta x}{8Q(1-V_y^2)} \qquad \left(\Delta V_x = \frac{-\Delta x}{\sqrt{2|\Delta x|Q/V_\infty}} \text{ for } V_y = 1\right)$$

We substitute $V_y = u/(1+\sqrt{1-u^2})$ into (3.8); taking into account that the external normal to the plate coincides with the negative direction of the x-axis, we find

$$\Delta V_n = -\Delta V_x = \frac{V_\infty}{Q}\frac{1}{\sqrt{2(1-u^2)}(1+\sqrt{1-u^2})^2}\times$$
$$\times\left\{1+\left[1+\left(\frac{V_\infty}{Q}\frac{u(1+3\sqrt{1-u^2})\Delta x}{(1-u^2)(1+\sqrt{1-u^2})^2}\right)^2\right]^{1/2}\right\}^{-1/2}\Delta x \qquad (3.9)$$

From boundary condition (3.1) and Equation (3.9) it follows that for harmonic oscillations of the plate the change of perturbed velocity with time does not conform to harmonic law. However, for harmonic oscillations of high frequency and small amplitude ΔV_n may turn out to be a quantity which is small to a higher order than $\partial\delta_n/\partial t$, then this term can be neglected in the boundary condition (3.1). From this it follows that for such oscillations changes of perturbed velocity on the plate conform to the harmonic law with a sufficient degree of accuracy.

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Translated by B.D.